ON STABILITY OF CERTAIN PERMANENT ROTATIONS OF A RIGID BODY

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We have found in [1] the permanent rotations of a rigid body with a fixed point in the force field of Goriachev. We investigate here the stability of such rotations.

1. We shall determine the position of a rigid body with a fixed point O by the direction cosines $(\alpha_i, \beta_i, \gamma_i)$ of the angles between the axes $x_1x_2x_3$ of the moving coordinate system fixed in the body and the nonmoving coordinate system $\xi\eta\zeta$. Here $i = 1, 2, 3$. The components of the angular velocity along the moving axes will be denoted by p_i, the direction cosines of the permanent axis by l_i, (with respect to moving axes) and the principal moments of inertia with respect to the moving axes by A_i. Let us consider the rigid body (in which $A_1 = A_2 = 2A_3$), acted on by forces whose potential has the form

$$U(\gamma_1, \gamma_2, \gamma_3) = a(n-1)^{-1}\gamma_3^{1-n} + {}^1/_2 b(\gamma_2^2 - \gamma_1^2) - c_1\gamma_1 - c_2\gamma_2 \qquad (1.1)$$

which corresponds to the attraction of four points of the body [1] by the fixed plane $\xi\eta$ according to the above formula. It is shown in [1] that the permanent axes move with a constant angular velocity through the arcs A_1S_1 and A_2S_2 if n is odd and through the arcs A_1S_1 and $A_2'S_2'$ when n is even. These spherical arcs are the intersections of two surfaces

$$l_1^2 + l_2^2 + l_3^2 = 1, \qquad c_2l_1 - c_1l_2 - 2bl_1l_2 = 0 \qquad (1.2)$$

with the perpendicular plane $\xi\eta$ (see Fig. in [1]). The points S_1, S_2 and S_2' correspond to the position of equilibrium $(\omega = 0)$, and A_1, A_2, A_2' determine the rotation with the angular velocity $\omega^2 = \infty$, which follows from Equations [1]

$$(b + {}^1/_2\omega^2)\, l_2l_3 + al_2l_3^{-n} - c_2l_3 = (b - {}^1/_2\omega^2)\, l_1l_3 - al_1l_3^{-n} + c_1l_3 = 0 \qquad (1.3)$$

We shall investigate the stability of the considered permanent axes with respect to the quantities p_i and γ_i. The components of the angular velocity of the body along the axes, moving with the body are

$$p_{i0} = \omega l_i \qquad (1.4)$$

In the perturbed motion we set

$$p_i = p_{i0} + \xi_i, \qquad \gamma_i = l_i + \eta_i \qquad (1.5)$$

The equations of the perturbed motion have the following first integrals:

$$V_1 = \sum_{i=1}^{3} (A_i\xi_i^2 + 2A_ip_{i0}\xi_i - 2a_i\eta_i) - \sum_{i,\,j=1}^{3} a_{ij}\eta_i\eta_j + O(\xi_i^3, \eta_i^3) = \text{const}$$

$$V_2 = \sum_{i=1}^{3} A_i(p_{i0}\eta_i + l_i\xi_i + \xi_i\eta_i) = \text{const}, \qquad V_3 = \sum_{i=1}^{3} (\eta_i^2 + 2l_i\eta_i) = 0 \tag{1.6}$$

where

$$a_i = (\partial U / \partial\gamma_i)_{\gamma_i = l_i}, \qquad a_{ij} = (\partial^2 U / \partial\gamma_i\partial\gamma_j)_{\gamma_i = l_i}$$

We shall construct the Liapunov function using the method of Chetaev [2] in the form of a linear combination of integrals (1.6)

$$V = V_1 - 2\omega V_2 + \lambda V_3$$

Here the constant λ has the form

$$\lambda = A_1\omega^2 + \frac{a_1}{l_1} = A_2\omega^2 + \frac{a_2}{l_2} = A_3\omega^2 + \frac{a_3}{l_3} =$$

$$= A_1\omega^2 - b - \frac{c_1}{l_1} = A_2\omega^2 + b - \frac{c_2}{l_2} = A_3\omega^2 - al_3^{\,n-1} \tag{1.7}$$

which comes from Euler's equations. The function V under the conditions (1.7) is

$$V = \sum_{i=1}^{3} [A_i\xi_i^2 - 2\omega A_i\xi_i\eta_i + (\lambda - a_{ii})\eta_i^2] - 2\sum_{i \neq j} a_{ij}\eta_i\eta_j + O(\xi_i^3, \eta_i^3) \tag{1.8}$$

It is obvious that the function V is positive definite with respect to ξ_i and η_i, if its quadratic part, that is $V - O(\xi_i^3, \eta_i^3)$ is positive definite.

By the Sylvester criterion the necessary and sufficient conditions for sign definiteness of this form are the inequalities

$$\lambda - a_{11} - A_1\omega^2 > 0, \qquad (\lambda - a_{11} - A_1\omega^2)(\lambda - a_{22} - A_2\omega^2) - a_{12}^2 > 0$$

$$\prod_{i=1}^{3} (\lambda - a_{ii} - A_i\omega^2) - \sum_{i=1}^{3} (\lambda - a_{ii} - A_i\omega^2)\, a_{i+1,\, i+2} - 2a_{12}a_{13}a_{23} > 0$$

For the function (1.1) these conditions become

$$-c_1/l_1 > 0, \qquad -c_2/l_2 > 0, \qquad -(n+1)\, al_3^{-n-1} > 0 \tag{1.9}$$

Under the conditions (1.9) the function V is the sign-definite integral of the equations of the perturbed motion, and by Liapunov's theorem on the stability, the nonperturbed motion (1.4) will be stable with respect to the variables p_i and γ_i.

The points on the arc $A_2'S_2'$ satisfy the sufficient conditions (1.9) and the arc is admissible when n is even.

We can show some unstable permanent axes when considering the linearized system of equations of the perturbed motion, where the therms of second and higher orders are neglected (with respect to the perturbations in the equations of the perturbed motion). The characteristic equation of the system has the form

$$\sigma^2(\sigma^4 + g_1\sigma^2 + g_2) = 0 \tag{1.10}$$

$$g_1 = (1 + {}^1/_4 l_3^2)\,\omega^2 - l_1(5bl_1 + 3c_1) - 3l_2(c_2 - {}^5/_3 bl_2) - al_3^{-n-1}(2l_3^2 + nl_1^2 + nl_2^2)$$

The expression for g_2 is not shown because it is too complicated.

The instability of the motion (1.4) occurs when the following inequalities are satisfied [3]

$$g_1 < 0, \qquad g_2 < 0, \qquad g_1^2 - 4g_2 < 0$$

For example, when the constants a, b, c_1 (shown in Fig. in [1]) are positive, then the inequality $c_2 > 2bl_2$ is satisfied. This follows from the construction of the branch of hyperbola (1.2) which passes through the origin. And then, obviously, $c_2 > {}^5/_3 bl_2$. Besides, on the admissible arcs A_1S_1

and A_2S_2 we have $l_1 > 0$ and $l_2 > 0$. Consequently, the second and the third terms in (1.10) will be negative. For the positive values o. n the term containing a in (1.10) will be negative, because $l_3 > 0$ on the arc A_1S_1, and the arc A_2S_2 will be admissible when n is odd.

Then the positions of equilibrium $\omega = 0$ determined by the points S_1 and S_2 will be unstable, since here $g_1 < 0$. The permanent axes passing through points near S_1 and S_2 will be unstable. The finite points of unstable arcs are determined by the values of ω^2 which make g_1 vanish.

2. Let a symmetrical body $(A_1 = A_2)$ be placed in the force field

$$U = aA_1 / (n-1)\,\gamma_3^{n-1} \qquad (n \neq 1, \quad a > 0)$$

The permanent axes with a constant velocity of rotation about them coincide with the ζ axis, and there are two such axes [1]

$$l_1 \neq 0, \quad l_2 \neq 0, \quad \omega^2 = a / (\varepsilon - 1)\, l_3^{n+1}, \quad \varepsilon = A_3 / A_1 \tag{2.1}$$

$$l_1 = l_2 = 0, \quad l_3 = 1, \quad \omega \text{ is an arbitrary} \tag{2.2}$$

We shall show now when the rotations (2.1) are unstable. The equations of the perturbed motion in this case are

$$(-1)^k \xi_k^{\boldsymbol\cdot} = (\varepsilon - 1)\,\omega\,[l_3(\xi_{3-k} - \omega\eta_{3-k}) + l_{3-k}(\xi_3 + n\omega\eta_3)] + O(\xi_i^2, \eta_i^2)$$

$$\eta_i^{\boldsymbol\cdot} = l_{i+1}(\xi_{i+2} - \omega\eta_{i+2}) - l_{i+2}(\xi_{i+1} - \omega\eta_{i+1}) + O(\xi_i^2, \eta_i^2) \tag{2.3}$$

$$\xi_3^{\boldsymbol\cdot} = 0 \qquad (k = 1,2; \ i = 1, 2, 3)$$

The characteristic equation of the linearized form of the system (2.3) is

$$\sigma^4(\sigma^2 + g) = 0$$

$$g = \omega^2 l_3^3 \left\{\left[\varepsilon - \frac{n}{2}\left(\frac{1}{l_3^2} - 1\right) - 2\right]^2 - \left(\frac{1}{l_3^2} - 1\right)\left[\frac{n^2}{4}\left(\frac{1}{l_3^2} - 1\right) + n - 1\right]\right\}$$

By the Liapunov's theorem on instability in the first approximation the unperturbed motion (2.1) is unstable when $g < 0$.

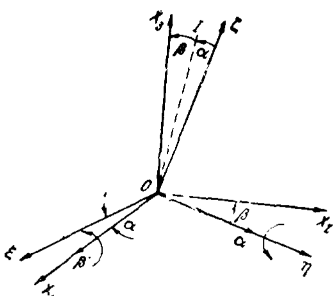

Fig. 1

When $n = 0$ (the case of Lagrange) the instability of the rotational motions (2.1) cannot be established from the characteristic equation of the linearized system. But when

$$2n > \left(1 + \frac{1}{|l_3|}\right)^{-1}, \qquad 2n < \left(1 - \frac{1}{|l_3|}\right)^{-1}$$

we have two real values

$$\varepsilon_1, \varepsilon_2 = {}^1/_2 n\,(l_3^{-2} - 1) + $$
$$+ 2 \mp \{(l_3^{-2} - 1)\,[{}^1/_4 n^2 (l_3^{-2} - 1) + n - 1]\}^{1/2}$$

and when $\varepsilon_1 < \varepsilon < \varepsilon_2$, the unperturbed motion (2.1) is unstable. Notice that when $\varepsilon > 0$, and the separation of the unstable motions (2.1) depends on the existence of $\varepsilon_2 > 0$. This takes place for example under the conditions

$$2\left(1 - \frac{1}{l_3^2}\right) < \frac{n}{2} < \left(1 - \frac{1}{|l_3|}\right)^{-1}$$

It is interesting that we can show the stability with respect to θ and $\theta^{\boldsymbol\cdot}$ $(\cos\theta = \gamma_3)$ when $g > 0$ by using Routh's theorem [4].

By the integrals (1.6) and also by the integral $V_4 = \xi_3 = \text{const}$, we can obtain the following condition for the stability of rotations (2.1) with respect to p_3 and γ_3.

Using Rumiantsev's method [5] we shall construct the Liapunov function

$$V = V_1 - 2\omega V_2 + \omega^2 V_3 + \varepsilon\mu V_4^2 = (\xi_1 - \omega\eta_1)^2 +$$
$$+ (\xi_2 - \omega\eta_2)^2 + (1 + \mu)\,\varepsilon\xi_3^2 - 2\varepsilon\omega\xi_3\eta_3 + \omega^2\,[1 - n(\varepsilon - 1)\,l_3^2]\,\eta_3^2 + O(\xi_i^3, \eta_i^3) \tag{2.4}$$

The constant $\mu > 0$ can be chosen such, that the quadratic part of the function (2.4) will be positive definite with respect to ξ_3 and η_3 if

$$1 - n(\varepsilon - 1)\, l_3^2 > 0 \tag{2.5}$$

Then by the theorem of stability with respect to some of the variables[5] the unperturbed motion (2.1) under the condition (2.5) is stable with respect to p_3 and γ_3.

The necessity of the sufficiency of the condition for the stability of motion (2.2) in the form

$$\varepsilon^2\omega^2 > 4a \tag{2.6}$$

can be proved by constructing the Liapunov function as the combination of Chetaev's integrals [2]

$$V = V_1 + 2\lambda V_2 - (a + \varepsilon\omega\lambda) V_3 + \mu V_4^2 - 2(\varepsilon\omega + \varepsilon\lambda) V_4$$

and taking as the Chetaev function $W = \xi_1\eta_1 - \xi_2\eta_2$, or using the condition of stability of rotations (2.2) obtained by Beletskii [6] for the force function $U = U(\gamma_3)$.

3. If the points of the rigid body are attracted by the stationary plane $\xi\eta$ proportionally to the distance from this plane, then the force function is $U = -\frac{1}{2}\mu\Sigma A_i\gamma_i^2$ and in the case of a dynamically symmetrical rigid body ($A_1 = A_2$) it takes the form $U = -\frac{1}{2}\mu(A_3 - A_1)\gamma_3^2 + \text{const}$. We shall have then the particular case of rotations (2.1) about the perpendicular line in the plane $\xi\eta$

$$a = \mu(\varepsilon - 1), \qquad n = -1, \qquad \omega^2 = \mu \tag{3.1}$$

The sufficient condition of stability (2.5) for these rotations takes the form

$$1 - l_3^2 + \varepsilon l_3^2 > 0 \tag{3.2}$$

When $l_3^2 < 1$ the conditions (3.2) are always satisfied, therefore the stability of the considered rotations occurs at any position of the body with respect to the perpendicular line in the plane $\xi\eta$. The sufficient conditions derived by Pozharitskii [7] allow to select only some stable motions (3.1).

4. We shall consider rotation of a symmetric rigid body acted on by a force which is constant in the moving coordinate system [1]. The body rotates about an axis which intersects this force, and we shall investigate the stability of this rotation with respect to the variables p_i. Equations of the perturbed motion

$$(-1)^k\xi_k^{\cdot} = (\varepsilon - 1)(p_{2+k}\xi_{k+1} - p_{k+1}\xi_{k+2} + \xi_{k+1}\xi_{k+2}), \quad \xi_3^{\cdot} = 0 \qquad (k = 1, 2) \tag{4.1}$$

have the general solution

$$\xi_1 = a\sin(\Omega t + \varphi) - bp_{10}(b + p_{30})^{-1}, \qquad \xi_2 = a\cos(\Omega t + \varphi) - bp_{20}(b + p_{30})^{-1}$$

$$\xi_3 = b, \qquad \Omega = (1 - \varepsilon)(b + p_{30}) \qquad (a, b, \varphi = \text{const}) \tag{4.2}$$

From (4.2) follows that the considered rotation is stable when $b \neq -p_{30}$.

5. If the points of a rigid body are attracted by the plane $\xi\zeta$ and are repulsed by the plane $\eta\zeta$ and the attraction is proportional to the distance from these planes then the force function is [8]

$$2U = \mu[A_1(\beta_1^2 - \alpha_1^2) + A_2(\beta_2^2 - \alpha_2^2) + A_3(\beta_3^2 - \alpha_3^2)] \tag{5.1}$$

The Euler-Poisson equations show that the permanent exis with the constant angular velocity of rotation exists in a symmertical body and it coincides with the ζ-axis.

The force function can be transformed into

$$2U = \mu(A_3 - A_1)(\beta_3^2 - \alpha_3^2) \tag{5.2}$$

which means that it depends only on the cosines of the angles between the axis of symmetry of the body and the fixed axes ξ and η. We shall investigate the stability of thid rotation. Considering the motion of the body in the variables p_1, α_1, β_1, γ_1, the equations of the perturbed motion which we obtain from the Euler-Poisson equations contain periodic functions, and the investigation of these equations even in the first approximation becomes difficult.

The problem simplifies, if the x_3-axis remains as before, while the x_1 and x_2 axes are arbitrarily located in the equatorial plane of the ellipsoid of inertia of the body and are not fixed in the body. The quantities α_3 and β_3 remain as before.

The position of the axis of symmery of the body x_3 is determined by the angle α between its projection I on the $\xi\eta$ plane and the ζ-axis (see Fig.1) and the angle β between the projection I and the x_3-axis itself. The amount of rotation of the body about the x_3-axis is denoted by the angle φ. The angles α, β, φ are the holonomic coordinates of the considered mechanical system. As a matter of fact, if we rotate the $\xi\eta\zeta$ system about the η-axis by the angle α it will occupy the position of the system $x_1\eta I$, and if we rotate the last system by the angle β about the x_1-axis it will occupy the position of the moving system $x_1x_2x_3$. Rotating the rigid body about the x_3-axis by the angle φ we obtain the given position of the body. By the well known theorem of analytic geometry on the cosine of an angle between two lines in space we have that

$$\alpha_3 = \cos x_3\xi = \sin\alpha\cos\beta, \qquad \beta_3 = \cos x_3\eta = -\sin\beta$$

From the proof that the α, β, φ coordinates are holonomic follows that the components of the angular velocity of rotation of the body along the moving axes are

$$p_1 = \beta^{\cdot}, \qquad p_2 = \alpha^{\cdot}\cos\beta, \qquad p_3 = \varphi^{\cdot} - \alpha^{\cdot}\sin\beta$$

Then the expression for the Lagrange function has the form

$$L = T + U = {}^1/_2 A_1(\alpha^{\cdot 2}\cos^2\beta + \beta^{\cdot 2}) + {}^1/_2 A_3(\varphi^{\cdot} - \alpha^{\cdot}\sin\beta)^2 + \\ + {}^1/_2\mu(A_3 - A_1)(\sin^2\beta - \sin^2\alpha\cos^2\beta)$$

The coordinate φ being cyclic corresponds to the following integral of the Lagrange equations

$$\partial L / \partial\varphi^{\cdot} = A_3 r_0 = A_3(\varphi^{\cdot} - \alpha^{\cdot}\sin\beta)$$

Ignoring the cyclic coordinates we obtain the equation in the Lagrange form with the Routh function

$$R = {}^1/_2 A_1(\alpha^{\cdot 2}\cos^2\beta + \beta^{\cdot 2}) + {}^1/_2 A_3 r_0^2 + {}^1/_2\mu(A_3 - A_1)(\sin^2\beta - \sin^2\alpha\cos^2\beta) - \\ - A_3 r_0(r_0 + \alpha^{\cdot}\sin\beta)$$

These equations do not contain the variable φ

$$A_1\alpha^{\cdot\cdot}\cos\beta - 2A_1\alpha^{\cdot}\beta^{\cdot}\sin\beta - A_3 r_0\beta^{\cdot} = -\mu(A_3 - A_1)\sin\alpha\cos\alpha\cos\beta$$
$$A_1\beta^{\cdot\cdot} + A_1\alpha^{\cdot 2}\sin\beta\cos\beta + A_3 r_0\alpha^{\cdot}\cos\beta = \mu(A_3 - A_1)(1 + \sin^2\alpha)\sin\beta\cos\beta \quad (5.3)$$

This permanent rotation leads to the following solution of the Equations (5.3)

$$\alpha = \beta = \alpha^{\cdot} = \beta^{\cdot} = 0, \qquad \varphi^{\cdot} = r_0 \quad (5.4)$$

In the perturbed motion we set

$$\alpha = \alpha, \quad \beta = \beta, \quad \alpha^{\cdot} = \alpha^{\cdot}, \quad \beta^{\cdot} = \beta^{\cdot}$$

Then the equations with variations have the form

$$A_1\alpha^{\cdot\cdot} + \mu(A_3 - A_1)\alpha - A_3 r_0\beta^{\cdot} = 0, \; A_1\beta^{\cdot\cdot} - \mu(A_3 - A_1)\beta + A_3 r_0\alpha^{\cdot} = 0 \quad (5.5)$$

The characteristic equation of the system (5.5) is

$$A_1\sigma^4 + A_3 r_0^2\sigma^2 - \mu^2(A_3 - A_1)^2 = 0$$

It always has positive roots, and, according to Liapunov's theorem on the instability in the first approximation, we deduce the instability of the motion (5.4).

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